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Rotating black holes in brane worlds

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Abstract: We study interaction of rotating higher dimensional black holes with a brane in space-times with large extra dimensions. In the approximation when a black hole is slowly rotating and the tension of the brane is small we demonstrate that the black hole loses some angular momentum to the brane. As a result of this effect a black hole in its final stationary state can have only those components of the angular momenta which are connected with Killing vectors generating transformations preserving a position of the brane. The characteristic time when a rotating black hole with the gravitational radius r_0 reaches this final state is $T \sim r_0^{p-1}/(G\sigma)$, where G is the higher dimensional gravitational coupling constant, σ is the brane tension, and p is the number of extra dimensions.

Keywords: Black Holes, Field Theories in Higher Dimensions, Models of Quantum Gravity.

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Contents

1. Introduction

There are several reasons why brane world scenarios [1, 2] became popular recently. One of the main reasons is that in these scenarios the fundamental energy scale can be as low as a few TeV which opens a possibility of the experimental tests of predictions of these models in the future collider and cosmic ray experiments [3, 4]. The most dramatic prediction of these models is a possibility of creation of mini black holes when the center of mass energy of two colliding particles becomes higher than a fundamental energy [3].

Being gravitational solitons black holes can propagate in the bulk space and thus to serve as probes of extra dimensions. We focus our attention on black holes with a size much smaller than the size of extra dimensions. In this case the effects of the extra dimensions on the black hole geometry are small and can be neglected. (For recent review of black holes in a space-time with large extra dimensions see [5].)

In a general case a black hole created by a collision of two particles is rotating. If there is no emission of the bulk gravitons the bi-plane of the black hole rotation lies within the brane. In a more realistic situation when bulk gravitons are emitted, a black hole can also acquire an angular momentum in the bi-plane not lying within the brane. There are other processes which may result in the black hole rotation in the bulk dimensions. For example, if a black hole collides with a particle or another black hole with emission of bulk gravitons, or when it emits bulk gravitons in the Hawking evaporation process.

The aim of this paper is to demonstrate that a rotating black hole interacting with a brane loses some of the components of its angular momenta. We study this effect in the approximation when a black hole is slowly rotating and the tension of the brane is small. We demonstrate that as a result of this friction effect a black hole in its final stationary state can have only those components of the angular momenta which are connected with Killing vectors generating transformations preserving a position of the brane. We illustrate this result first by considering a 4-dimensional Kerr black hole interacting with a thin domain wall. Next we prove this result for a slowly rotating higher dimensional black holes interacting with branes.

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Our calculations show that the characteristic time of the relaxation during which a rotating black hole reaches the equilibrium state is shorter than the time during which it loses its bulk angular momentum because of the Hawking radiation. This may have important experimental signature of mini-black holes in future collider and cosmic ray experiments.

2. Static black holes

In the approximation when the gravitational back-reaction of the brane is neglected its world-sheet obeys the Nambu-Goto equation

$$ {}^{(n+1)}\Delta X^\mu + \gamma^{ab}\Gamma^\mu{}_{\nu\lambda} X^\nu_{,a} X^\lambda_{,b} = 0\,. \tag{2.1} $$

The relations $X^\mu = X^\mu(\zeta^a)$ determine the embedding of the $n+1$-dimensional brane into $N+1$-dimensional bulk space-time. ζ^a, $(a, b = 0, n)$, are internal coordinates in the brane and X^μ, $(\mu = 0, N)$ are coordinates in the bulk space with the metric $g_{\mu\nu}$. The connections $\Gamma^\mu{}_{\nu\lambda}$ are determined for $g_{\mu\nu}$. To exclude degenerate cases we assume that $0 < n < N$. ${}^{(n+1)}\Delta$ is the box-operator for the induced metric

$$ \gamma_{ab} = g_{\mu\nu}\, X^\mu_{,a} X^\nu_{,b}\,. \tag{2.2} $$

The stress-energy tensor of the brane is defined as follows:

$$ \sqrt{-g}T^{\mu\nu} = \sigma \int d^{n+1}\zeta\, \delta^{(N+1)}(X - X(\zeta))\, t^{\mu\nu}\,, \tag{2.3} $$

where $t^{\mu\nu} = \sqrt{-\gamma}\gamma^{ab} X^\mu_{,a} X^\nu_{,b}$ and σ is the brane's tension.

The metric of a non-rotating higher dimensional black hole is [6]

$$ d\bar{s}^2 = -B dv^2 + 2 dr dv + r^2 d\Omega^2_{N-1}\,. \tag{2.4} $$

The coordinate v is the advanced time, $d\Omega^2_{N-1}$ is the metric on the unit sphere S^{N-1}, and $B = 1 - (r_0/r)^{N-2}$. For $N = 3$ this metric reduces to the Schwarzschild metric. The gravitational radius r_0 is related to the black hole mass M as follows

$$ M = \frac{(N-1)\mathcal{A}_{N-1}}{16\pi G_{N+1}} r_0^{N-2}\,, \tag{2.5} $$

where $\mathcal{A}_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}$ is the area of a unit sphere S^{N-1} and G_{N+1} is the $N+1$-dimensional gravitational coupling constant which has dimensionality $[length]^{(N-2)}/[mass]$.

Consider a unit sphere S^{N-1} embedded in a N-dimensional euclidean space R^N, and let X^A, $(A = 1, \ldots, N)$ be the cartesian coordinates in R^N. One can choose these coordinates so that the equations $X^{n+1} = \cdots = X^N = 0$ determine the n-dimensional hyper-surface (brane). This hyper-surface intersects the unit sphere S^{N-1} along a surface $\mathcal{S}$ which has a geometry of a round unit sphere S^{n-1}. The surface $\mathcal{S}$ is a higher dimensional analogue of a 'large circle' on a two-dimensional sphere. In particular, being considered as a sub-manifold of S^{N-1} it has a vanishing extrinsic curvatures, and hence is a geodesic sub-manifold. We denote by ω^α coordinates on $\mathcal{S}$, and by $d\omega^2_{n-1}$ the metric on it.

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One can construct a solution for a static n-brane as follows. We use $\zeta^a = (v, r, \omega^\alpha)$ as coordinates on the brane. Then

$$d\bar{\gamma}^2 = -Bdv^2 + 2dr\,dv + r^2 d\omega_{n-1}^2\,, \tag{2.6}$$

is the induced geometry on the brane. It is easy to check that such a surface is geodesic and hence is a solution of the Nambu-Goto equations (2.1).

Denote by ξ^μ a Killing vector field generating a rotation in some bi-plane. Then the flux per unit time v of the corresponding angular momentum of the brane, $\dot{J}^b$, through the surface r=const is given by the expression

$$\dot{J}^b = -\int_{r=\mathrm{const}} \sqrt{-g} T^{r\nu} \xi_\nu d^{N-1}\Omega\,. \tag{2.7}$$

Due to the conservation law $T^{\mu\nu}{}_{;\nu} = 0$ the flux $\dot{J}^b$ does not depend on r. Let us denote by $\dot{J}$ the rate of the loss of the angular momentum of the black hole. The angular momentum is transmitted to the brane and therefore $\dot{J} = -\dot{J}^b$. By using (2.3), (2.7) one finds

$$\dot{J} = \sigma \int_{r=\mathrm{const}} d^{n-1}\omega\, t^{\mu\nu} n_\mu \xi_\nu\,, \tag{2.8}$$

where $n_\mu = r_{,\mu}$. The integral is over $(n-1)$ dimensional sphere and $d^{n-1}\omega$ is a measure on a unit sphere S^{n-1}.

For a static black hole, since ξ^μ is tangent to the surface r=const, $\dot{J} = 0$.

3. Kerr black hole

Before considering higher dimensional rotating black holes let us discuss a simpler case of a brane attached to the rotating black hole in the 4-dimensional space-time. We consider only slowly rotating black holes. For $a/M \ll 1$ one can write the Kerr metric in the form

$$ds^2 = d\bar{s}^2 - 2a \sin^2\theta\, d\varphi \left(\frac{r_0}{r}\, dv + dr\right). \tag{3.1}$$

Here $d\bar{s}^2$ is the Schwarzschild metric

$$d\bar{s}^2 = -Bdv^2 + 2dr\,dv + r^2(\sin^2\theta d\varphi^2 + d\theta^2)\,, \tag{3.2}$$

and $B = 1 - r_0/r$, $r_0 = 2M$. Denote by α an angle between the axis of rotation and the brane, then the equation of the unperturbed domain wall is $\varphi = \bar{\varphi}(\theta)$ where

$$\sin\bar{\varphi} = \tan\alpha\, \cot\theta\,, \tag{3.3}$$

and $\alpha \le \theta \le \pi - \alpha$ for $0 \le \alpha \le \pi/2$. The induced metric on the world-sheet of such a tilted domain wall is

$$d\bar{\gamma}^2 = -Bdv^2 + 2dr\,dv + \frac{r^2 \sin^2\theta}{\sin^2\theta - \sin^2\alpha} d\theta^2\,. \tag{3.4}$$

The domain wall deformed by the black-hole rotation is described by the equation $\varphi = \bar{\varphi} + \psi$, where ψ obeys the equation

$$^{(3)}\bar{\Delta}\psi + \frac{2}{r^2}\cot\theta\psi_{,\theta} + 2\frac{B}{r}\psi_{,r} = \frac{a}{r^3}\,, \tag{3.5}$$

where $^{(3)}\bar{\Delta}$ is the box-operator in the metric (3.4). This equation has a solution $\psi = -a/r$. It is possible to show that this is a unique solution which is regular both at the horizon and infinity [8].

For this regular solution, T^r_φ is given by the following expression

$$\sqrt{-g}T^r_\varphi = -\sigma r_0\, a\sin\theta\sqrt{\sin^2\theta - \sin^2\alpha}\,\delta(\varphi - \bar{\varphi}(\theta))\,. \tag{3.6}$$

This quantity is already of the first order in a and one can use (2.7) to obtain

$$\dot{J} = -\pi\sigma a r_0\cos^2\alpha = -2\pi G_4\sigma\cos^2\alpha J\,. \tag{3.7}$$

The angular momentum flux vanishes when the domain wall is in the equatorial plane of the rotating black hole.[1] Thus this is the final stationary equilibrium configuration of the rotating black hole in the presence of the domain wall. The relaxation time when the black hole reaches this final state is $T \sim (\pi G_4\sigma\cos^2\alpha)^{-1}$.

For a cosmic string attached to the Kerr black hole, a similar problem can be solved for an arbitrary value of the rotation parameter a because solutions of the Nambu-Goto equations are known exactly [7]. In this case the final stationary configuration is a string directed along the rotation axis. The relaxation time is $T \sim r_0/(4G_4\sigma\sin^2\alpha)$, where α is an initial angle between the the string and the axis of rotation [8].

4. Higher dimensional rotating black holes

Now we consider a general case. We assume that a N-dimensional rotating black hole is attached to a n-dimensional brane. If the black hole size is much smaller that the size of extra dimensions, and the tension of the brane is small, the gravitational field of the black hole is described by the Myers-Perry (MP) metric [9].[2] This metric besides the time-like at infinity Killing vector $\xi^\mu_{(t)}$ has $[N/2]$ (the integer part of $N/2$) mutually commutative and mutually orthogonal Killing vectors $\xi^\mu_{(i)}$ singled out by the property that they have closed integral lines. The Killing vectors $\xi_{(i)}$ are elements of the Cartan sub-algebra of the group of rotations SO(N). The MP metric is characterized by the gravitational radius r_+ and by $[N/2]$ rotation parameters a_i. Such a black hole has angular velocities $\Omega_i = a_i/(r_+^2 + a_i^2)$. The vector $\eta = \xi_{(t)} + \sum_i \Omega_i\xi_{(i)}$ on the horizon becomes a null generator of the horizon. (The summation over i is performed from $i = 1$ to $i = [N/2]$.)

[1]It is easy to check that for an arbitrary value of a the equatorial plane of the Kerr metric is a geodesic surface and hence it obeys the Nambu-Goto equations. For this configuration there is no angular momentum transfer

[2]In the higher dimensional case there is no uniqueness theorem and there may exist rotating black hole solutions different from the MP metric. An example is a black ring solution [10]. This metric has no slow rotation limit. For this reason we do not discuss it in the paper. Also in the slow rotation limit the effect of instability of rapidly rotating black holes [11] is not important

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For slow rotation $a_i/r_0 \ll 1$, the MP metric in the Kerr-incoming coordinates takes the form

$$
\begin{aligned}
ds^2 &= d\bar{s}^2 - \frac{2}{r^2}\left[dr + \left(\frac{r_0}{r}\right)^{N-2} dv\right] \varrho_\mu dx^\mu \,, && (4.1)\\
\varrho^\mu &= \sum_i a_i \xi^\mu_{(i)} \,, && (4.2)
\end{aligned}
$$

where $d\bar{s}^2$ is the unperturbed metric (2.4). For this form of the metric relations $\mathcal{L}_{\xi_{(i)}}\bar{g}_{\mu\nu} = 0$ imply that $\mathcal{L}_{\xi_{(i)}} g_{\mu\nu} = 0$. The angular momenta of the black hole J_i are defined as

$$
J_i = \frac{\mathcal{A}_{N-1}}{8\pi G_{N+1}} r_0^{N-2}\, a_i = \frac{2}{N-1} M a_i \,. \tag{4.3}
$$

In the linear approximation (4.1) $r_+ \approx r_0$, $\Omega_i \approx a_i/r_0^2$, and $\eta = \xi_{(t)} + r_0^{-2}\varrho$.

Consider a static brane in the metric $d\bar{s}^2$. We analyze now what happens with the brane in the presence of slow rotation. Interaction of the brane with the black hole results in the change of the brane position $X^\mu = \bar{X}^\mu + \delta X^\mu$. By linearizing the Nambu-Goto equations (2.1) one obtains a linear equation for δX^μ. It is possible to show that a solution of these equations which is regular at the horizon and infinity is [8]

$$
\delta X^\mu = \psi(r)\, \varrho^\mu \,, \qquad \psi'(r) = \frac{(1 - (r_0/r)^{n-1})}{(r^2 B)} \,. \tag{4.4}
$$

We calculate now the rate of the loss of the angular momentum of the black hole which interacts with a stationary brane. Since for the static metric $d\bar{s}^2$ $\dot{J}_i$ vanishes, it is sufficient to calculate the variation of (2.8) induced by the metric perturbations. We have

$$
\dot{J}_i = \sigma \int_{r=\mathrm{const}} d^{n-1}\omega\, \delta(t^{\mu\nu} n_\mu \xi_{(i)\nu}) \,. \tag{4.5}
$$

It is easy to check that in the linear in a_i approximation the following relations for the variations induced by the perturbed metric (4.1) are valid: $\delta(\sqrt{-\gamma}) = 0$,

$$
\begin{aligned}
\xi_{(i)a}\delta\gamma^{ra} &= \frac{1}{r^2}\,(\xi^{\|}_{(i)}, \varrho^{\|}) - B(\xi^{\|}_{(i)}, \delta X^{\|}_{,r}) \,, && (4.6)\\
\gamma^{ra}\delta\xi_{(i)a} &= -\frac{a_i}{r^2}\,(\xi_{(i)})^2 \,, && (4.7)\\
\gamma^{ra}\,\xi_{(i)\lambda}\delta X^\lambda_{,a} &= B(\xi_{(i)}, \delta X_{,r}) \,. && (4.8)
\end{aligned}
$$

We denote by $p^{\|}$ a projection of the vector p on the brane. (p,q) is a scalar product of vectors p and q in the unperturbed metric, $(p,q) = \bar{g}_{\mu\nu} p^\mu q^\nu$.

The flux of the angular momentum from the black hole to the brane changes the angular momenta of the black hole (4.3). In the linear approximation the equations for the change of the angular momenta of the black hole can be written as follows

$$
\dot{\mathbf{J}} = -\mathbf{K}\,\mathbf{J} \,. \tag{4.9}
$$

We use bold-faced quantities for vectors and tensors in the space of rotation parameters, so that $\mathbf{J}$ and $\mathbf{K}$ have components J_i, and K_{ij}. Relations (4.3)–(4.8) enable one to get $\mathbf{K}$ in the form

$$K_{ij} = \frac{(N-1)\sigma r_0^{n-1} k_{ij}}{(2M)}, \tag{4.10}$$

$$k_{ij} = \int_{S^{n-1}} d\omega^{n-1} \frac{1}{r^2} (\xi^{\perp}_{(i)}, \xi^{\perp}_{(j)}) . \tag{4.11}$$

We denote by $p^{\perp}$ a projection of a vector p orthogonal to the brane. In an agreement with the conservation law, the angular momentum flux does not depend on the radius r of the surface where it is calculated.

Note that in the linear in a_i approximation $\dot{M} = \dot{r}_0 = 0$ and M and r_0 in (4.10) are considered as constant parameters. The evolution equation (4.3) can also be written as $\dot{\mathbf{a}} = -\mathbf{K}\,\mathbf{a}$, where $\mathbf{a}$ is a vector with components a_i. This equation shows that the black hole can be stationary, $\dot{\mathbf{a}} = 0$, if and only if $\mathbf{a}$ is the zero vector, $\mathbf{Ka} = 0$. In this case the equation $\mathbf{a}^T\mathbf{Ka} = 0$ implies that

$$\int_{S^{n-1}} d\omega^{n-1} (\varrho^{\perp}, \varrho^{\perp}) = 0 , \tag{4.12}$$

and hence $\varrho^{\perp} = 0$. This means that the corresponding Killing vector ϱ generates transformations that preserve a position of the brane. The stationary metric of the final black hole configuration in this case is given by (4.1) where ϱ is a vector tangent to the brane. Because in the considered approximation ϱ is related to the null generator of the black hole horizon, $\eta = \xi_{(t)} + r_0^{-2}\varrho$, one can also describe the final state of the black hole as a state where η is tangent to the brane world-sheet.

From (4.4) it follows that in the final stationary state $\delta X^{\perp\mu} = 0$. Since the tangent to the brane components of δX^{μ} can always be gauged away, the brane in this case is not deformed.

Besides zero eigenvectors, the non-trivial matrix $\mathbf{K}$ has eigenvectors with positive eigenvalues. These eigenvectors define the directions in the space of parameters a_i for which the evolution is damping. The damping is caused by the 'friction' which is a result of the interaction between the black hole and the brane. The characteristic time of the relaxation process during which the black hole reaches its final state is

$$T \sim \frac{r_0^{p-1}}{(G_{N+1}\sigma)} \sim T_* \left(\frac{r_0}{L_*}\right)^{p-1} \left(\frac{\sigma_*}{\sigma}\right) . \tag{4.13}$$

Here $p = N - n$ is the number of extra dimensions, $\sigma_* = M_*/L_*^n$ and quantities M_*, L_* are, respectively, the fundamental mass and length of the theory.

The black hole can also lose its bulk components of the rotation by emitting Hawking quanta in the bulk. The characteristic time of this process is $T_H \sim T_*(r_0/L_*)^N$. For black holes which can be treated classically $r_0 \gg L_*$, $T_H \gg T$. Thus the friction effect induced by the brane is the dominant one.

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5. Discussion

We considered interaction of rotating black holes with branes. Such systems include several physically interesting examples, such as cosmic strings and thin domain walls interacting with the Kerr black hole, as well as rotating black holes in a brane world. In the slow rotation approximation we demonstrated that there exist an angular momentum transfer from a black hole to the attached brane. It vanishes when the generator of rotation ϱ, (4.2), is tangent to the brane. One can expect that a similar result may be valid beyond the adopted approximation, that is when a black hole is not slowly rotating and the brane generates a nontrivial gravitational field. It will happen for example if a higher dimensional analogue of the Hawking theorem [12] is valid.

We would like to conclude the paper by the following remark. We focused our attention on the higher dimensional space-times with vanishing bulk cosmological constant (ADD model [1]). A similar problem concerning general properties of higher dimensional rotating black holes with the horizon radius r_+ can be addressed in the Randall-Sundram (RS) models [2] provided the bulk cosmological constant is much smaller than r_+^{-2}. A characteristic property of such models is the existence of Z_2 symmetry. Under Z_2 transformation the brane remains unchanged, while the components of any vector orthogonal to the brane change their sign. Thus Z_2 symmetry implies $\varrho^\perp = 0$. Hence a stationary black hole attached to the brane in the RS-model can rotate only within the brane. Exact solutions describing rotating black holes on two-branes [14] possess this property.

The relaxation process related to the presence of $\varrho^\perp$ which is typical for the ADD-model is absent in the RS-model. This is an additional signature which in principle may allow one to distinguish between these models in observations.

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